# Newtonian Mechanics: An Implication of Extended Relativity

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**Abstract.** This paper asserts that the set of basic laws of classical mechanics is an inductive implication of extended principle of relativity – Galilean relativity plus relativity of scale.

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## INTRODUCTION

Here we take a look at the foundations of classical mechanics from somewhat different perspective in an attempt to show that the laws of Newton and the law of universal gravitation all follow from a single principle. That sounds like a farfetched idea, but as John E. Littlewood has noted[1]: "*Erasmus Darwin held that every so often you should try a damn-fool experiment. He played the trombone to his tulips. This particular result was in fact negative*." Darwin's rule, notwithstanding its humorous attire, is worth to be taken quite seriously.

## KINETIC ENERGY AND GALILEAN PRINCIPLE OF RELATIVITY

One of the fundamental notions of classical mechanics is that of kinetic energy. Every high school student knows that kinetic energy of a moving body equals half the product of mass and velocity squared: $K = mv^2/2$. This functional relation is a direct and simple corollary of the second law of Newton.

Is it feasible to derive the formula for kinetic energy without appealing to Newton's second law? What is kinetic energy to begin with? Lacking a precise definition, one could start with an obvious observation that kinetic energy is associated with the capacity of a moving body to inflict some "damage", or enact some "change" in the state of another body at impact solely due to the fact that it is moving as a *whole*. Daily experience suggests that this capacity – live force as Leibniz called it – is a monotonically increasing function of both mass and velocity. Ruling out a priori the possibility that kinetic energy might vary with other parameters like volume, temperature, shape, etc. is perhaps questionable, but functional relation $K = F(m,v)$ seems, at least, a reasonable heuristic conjecture.

# Interaction of Two Identical Bodies

Here we do not engage in a detailed discussion of what inertial mass is; we simply take it as an *additive* measure of material body's ability to resist changing its state of motion. Let us fix the speed of a moving body and try to work out the functional relation between material object's kinetic energy and its mass: $K = F(m,v_0) = g(m)$.

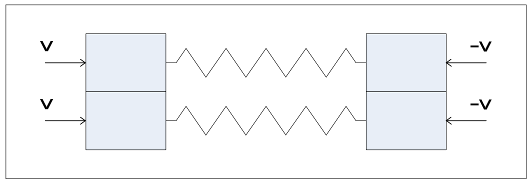


**FIGURE 1**

If two identical bodies moving with the same absolute speed from opposite directions along a horizontal, flat, and frictionless surface can compress an elastic spring to a certain degree before coming to a complete stop, then two such pairs of bodies evidently have the capacity of compressing two such springs to the same degree. This is a simple symmetry argument. Hence, due to assumed additivity of inertial mass,

$$2g(m) = g(2m) \;\rightarrow\; g^{I}(m) = g^{I}(2m) \;\rightarrow\; g(m) \sim m; \qquad (1.1)$$

i. e. kinetic energy of a moving body is proportional to its mass: $K = mf(v)$.

Galilean principle of relativity, as we show next, imposes certain restrictions on the function $f(v)$. Imagine two identical bodies moving with the same speed $w$ along a flat, horizontal, and frictionless surface; between the bodies, there is a compressed massless elastic spring prevented from decompressing by a thread (Fig. 2).

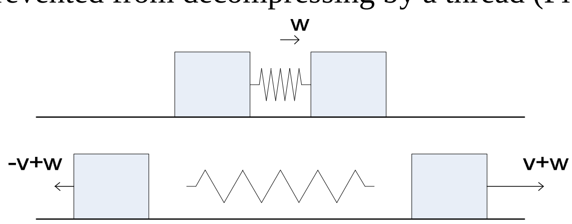


**FIGURE 2**

Cutting the thread releases the spring thereby changing kinetic energy of the system:

$$\Delta K = mf(-v+w) + mf(v+w) - 2mf(w). \qquad (1.2)$$

Seeing different values of $w$ as velocities of different inertial reference frames, we get a restraining condition on $f$ imposed by Galilean principle of relativity:

$$f(-v) + f(v) = f(-v+w) + f(v+w) - 2f(w). \qquad (1.3)$$

Assuming double differentiability of $f$ and differentiating (1.3) by $v$ then by $w$, we get:

$$f^{II}(-v+w) = f^{II}(v+w). \qquad (1.4)$$

The general solution of this equation, satisfying the condition $f(0) = 0$, is:

$$f(v) = av^2 + bv. \qquad (1.5)$$

Thus, using nothing but Galilean principle of relativity and elementary symmetry arguments, we have managed to reduce conjectured functional relation $K = F(m,v)$ to

$$K = m(av^2 + bv). \qquad (1.6)$$

# Vis Viva Controversy

Symmetry arguments will prove invaluable in narrowing down the expression (1.6) even further. On the one hand, since kinetic energy is associated with velocity, it would seem reasonable to argue that it should be defined and treated as a vector quantity. On the other hand, there is no reason to believe that a horizontally flying object has more capacity to inflict "damage" compared to the same object flying at the same speed in the opposite direction, therefore, $|m(av^2 + bv)| = |m(av^2 - bv)|$. For this equality to hold either $a$, or $b$ must be necessarily zero, that is either $K \sim mv$, or $K \sim mv^2$.

Which one is correct? That was precisely the subject of the famous debate that started in the 17th century and became known in the history of science as *vis viva controversy*. Descartes (1633 and 1644), and later Newton, had argued that *vis viva* is the product of mass and velocity. Leibniz objected to this (1686) that experiments with falling and rising bodies indicate *vis viva* to be another quantity: the product of mass and velocity squared. Although today it seems hard to understand how a concept like kinetic energy could be controversial, the heated exchange drew other illustrious scholars in, and the dispute lasted for almost 100 years. It is not difficult, though, to demonstrate that conjecture $K \sim mv$ contradicts Galilean principle of relativity.

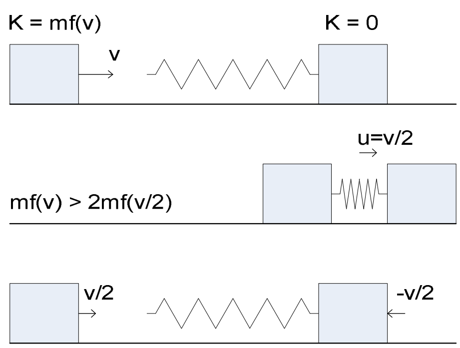


**FIGURE 3**

At the top of Fig. 3 we see two identical bodies on a horizontal and frictionless flat surface; one is moving with velocity $v$ and the other is at rest. As soon as the moving body hits the elastic spring it starts slowing down, and the resting body starts accelerating. At some point – at the point of maximum spring compression to be exact – velocities of the bodies get equal and, for a split of a second, they'll be moving with the same speed $u = v/2$. Indeed, in a reference frame moving from the left to the right with velocity $v/2$, perfect symmetry in the motion of identical bodies becomes apparent; therefore, exactly at the moment of maximum spring compression, both bodies will come to rest. Switching back to the original frame of reference we get $u=v/2$. Prior hitting the spring, kinetic energy of the moving body was $K = mf(v)$. At the point of maximum spring compression, share of this energy, specifically $mf(v/2)$, has been retained by the moving body; equal share of the energy has been passed to the other body; yet another share went to changing the spring state from uncompressed to that of maximum compression. Without resorting to the law of energy conservation (note that we haven't defined the notion of potential energy; indeed, we don't even have to for our purposes!), we cannot tell exactly what portion of kinetic energy has been wasted to “damage” the spring. Nevertheless, there can be no doubt that

$$mf(v) > 2mf(v/2) \tag{1.7}$$

Inequality (1.7) is obviously impossible with $f(v) \sim v$; so we have only one option left, namely $f(v) \sim v^2$, with the corresponding final expression for the kinetic energy:

$$K = kmv^2 \tag{1.8}$$

## Principal Schema for Experimental Testing

We have arrived at inescapable inductive logical inference: Galilean relativity implies $K = kmv^2$. Is there a simple and convincing way to test this experimentally? It is important to note that we haven’t defined any methodology for comparing *distinct* “damages” caused by distinct sources of kinetic energy. But that is not necessary at all for experimental verification of the formula for kinetic energy. Indeed, suffice to confirm that a body of mass $m$ moving with velocity $v$ has a capacity to inflict the *same* “damage” as the body of mass $2m$ moving with velocity $v/\sqrt{2}$.

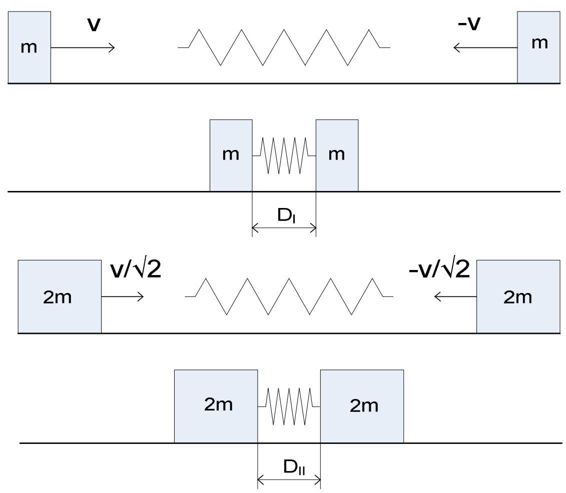


**FIGURE 4**

If two pairs of identical bodies moving symmetrically along a horizontal flat surface with no friction, as shown in Fig.4, do compress two identical elastic springs to the same degree, i.e. if $D_I = D_{II}$, then a logical conclusion follows that a body of mass $m$ moving with velocity $v$ has the same amount of kinetic energy as the one of mass $2m$ moving with velocity $v/\sqrt{2}$. That constitutes a conclusive experimental argument in favour of the derived above formula for kinetic energy $K = kmv^2$.

## Newton’s Second Law

As noted in section 1, formula for kinetic energy follows from the second law of Newton. Having logically derived the expression for kinetic energy $K = kmv^2$ from Galilean principle of relativity alone, we can now turn things around and derive Newton’s second law from it. Constant factor $k$ can be fixed arbitrarily without affecting the essence of physical theory. In Newtonian mechanics $k = 1/2$.

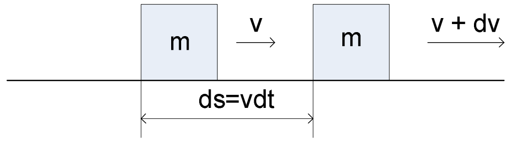


**FIGURE 5**

Let a body of mass *m* moving with some speed *v* along a frictionless, flat and horizontal surface be subjected to a force in the direction of movement for a short period of time ***dt*.** The action of force changes body's position and velocity as shown in Fig.5. For the kinetic energy differential we have

$$dK = m(v+dv)^2/2 - mv^2/2 = mvdv. \qquad (1.9)$$

Recalling the definitions of force ($F = dK/ds$) and velocity ($v = ds/dt$) Newton's second law follows from (1.9) instantly:

$$F = mdv/dt \qquad (1.10)$$

# CONSERVATION OF MOMENTUM AND GALILEAN RELATIVITY

We are not done with the implications of the Galilean principle of relativity yet. The conservation of linear momentum, as we shall prove next, is also a direct corollary of this principle.

## Interaction of Two Distinct Bodies

We apply now Galilean principle of relativity to the interaction of two objects of different inertial mass to see where it leads.

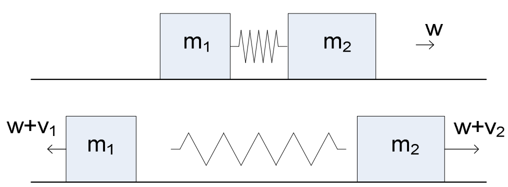


**FIGURE 6**

Just like in the case of interaction of two identical bodies, releasing the compressed spring leads to a change of kinetic energy in the system (Fig.6):

$$\Delta K = km_1(w+v_1)^2 + km_2(w+v_2)^2 - k(m_1+m_2)w^2 \qquad (2.1)$$

The magnitude of this change is the same in all inertial reference frames, therefore:

$$km_1v_1^2 + km_2v_2^2 = km_1(w+v_1)^2 + km_2(w+v_2)^2 - k(m_1+m_2)w^2 \qquad (2.2)$$

Past simple algebra, (2.2) yields the law of conservation of linear momentum:

$$m_1v_1 + m_2v_2 = 0 \qquad (2.3)$$

It shall be noted that equation (2.3) has been derived for solid bodies, i.e. when all parts of each interacting body move with the same speed. It is interesting to see what happens when one of the bodies is solid and the other is made of *N–1* identical and separated from each other solid pieces in a solid container. The container, which is kind of a black box, has mass equal to the mass of each piece it contains.

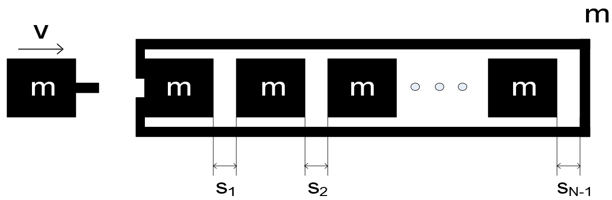

**FIGURE 7**

Another body of mass $m$, moving originally at speed $v$, comes to a complete halt after hitting our black box, and the container starts jumping like a frog at equal time intervals $\Delta t=(S_1+S_2+\ldots+S_{N-1})/v$. First it will jump by $S_1$, then by $S_2$, ... , then by $S_{N-1}$. It will take time interval of $(N-1)\Delta t+\Delta t=N\Delta t$ for the container to advance by $S=(S_1+S_2+\ldots+S_{N-1})$. Therefore the average velocity of the black box, $V=S/(N\Delta t)=v/N$, does not depend on $\Delta t$. The total impulse of the system before the collision was equal to $mv$. After the collision, the total mass of the black box, $mN$, multiplied by the average velocity of the container, $v/N$, yields the same impulse $mv$, i.e. the impulse of the black box calculated in this way is also conserved. That average velocity $V$ is, obviously, the velocity of the center of mass of the black box.

This may appear a bit paradoxical. Indeed, by taking to the limit $\Delta t \to 0$, it seems that black box advance can be made as smooth as desired. The total kinetic energy of the black box then would be $(mN)V^2/2 = (mv^2/2)/N$, i.e. it looks like we have lost most of the original kinetic energy $mv^2/2$! This “paradox” has a simple explanation: no matter how “smooth” the advance of the black box as a whole might appear to the naked eye, only small part of it is actually moving at any given point of time.

## Newton’s Third Law

The law of linear momentum conservation (2.3) has been derived without making any specific assumption about the character of the interaction – it could be via a spring, an explosion, etc. Neither did we specify the intensity of the interaction.

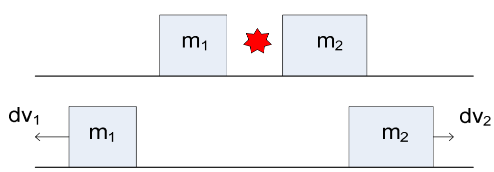

**FIGURE 8**

In the case of an interaction taking an infinitesimal period of time $dt$ (Fig. 8), the algebraic equation (2.3) becomes a differential one:

$$m_1dv_1 + m_2dv_2 = 0 \qquad (2.4)$$

A simple division of this equation by $dt$ leads to Newton’s third law (for every action there is an equal and opposite reaction):

$$m_1dv_1/dt = -m_2dv_2/dt \quad \to \quad F_1 = -F_2 \qquad (2.5)$$

# The Law of Universal Gravitation and Relativity of Scale

The law of universal gravitation asserts that every object in the universe attracts every other object with a force which for any two bodies is proportional to the mass of each and varies inversely as the square of the distance between them: $F = Gm_1m_2/r^2$. Newton was not the first to conjecture the law of gravitation, but he was the first to prove it with mathematical rigor by showing that Kepler's laws, derived empirically from the experimental observations of Tycho Brahe, follow from inverse square law.

As Feynman has pointed out[2], Newton made no hypotheses about the machinery behind the phenomenon of gravitation; he was satisfied to find what gravitation did without getting into the machinery of it.

We know that the force of electrical attraction and repulsion between charged particles also varies inversely as the square of separation of the particles. Is this remarkable similarity between gravitational and electrical forces a mere coincidence? Are there some basic principles from which such behaviour of gravitational and electrical forces would naturally follow?

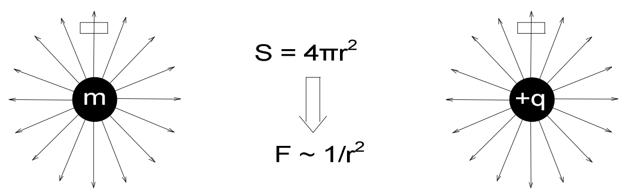


**FIGURE 9**

Since surface area surrounding any gravitational or electrical charge is increasing as square of distance, it seems natural to see inverse square law as mathematical expression for some kind of influx conservation law (Fig.9). The problem with this is that neither classical mechanics nor classical electrodynamics is ready to deal with the question of reality of an intermediary physical substance responsible for the machinery of the interaction. Here we offer a theoretical argument of entirely different nature – the relativity of scale which is not to be confused with Galilean relativity. It does not make sense to speak of material body's motion unless one specifies another body relative to which it is moving. On par with this logic, it does not make sense to speak of material object's size, or the distance between two such objects, unless a measuring rod is specified. This is a different type of relativity – relativity of scale. Now, following Edwin Jaynes, "if we adopt – almost surely true – hypothesis that our allegedly 'elementary' particles cannot occupy mere mathematical points in space, but are extended structures of some kind"[3], then, as we show next, the law of universal gravitation follows from this principle of relativity of scale, just like the laws of Newton follow from Galilean principle of relativity.

Let us imagine that distances between all celestial bodies in the solar system as well as their sizes and velocities, including the sizes and distances between all their constituent parts ('elementary' particles) at each and every level of fractal structure of matter, have been reduced, or enlarged instantaneously by the factor of $k$ (Fig. 10).

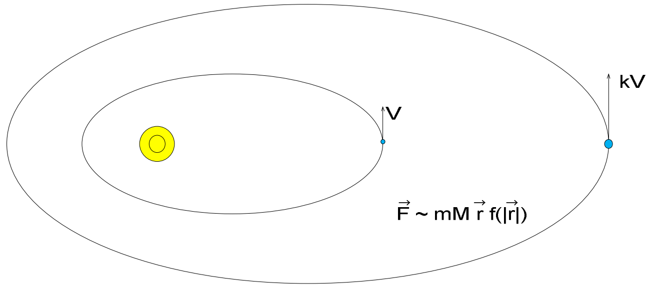


**FIGURE 10**

Would such scale transformation alter the evolution of the solar system? In other words, would the sequence of observable configurations of the solar system change? The answer is *no* – it would be impossible to tell the difference without "looking out the window". And that is precisely because the law of universal gravitation is the way it is. Replace the inverse square law with any other law, and scale invariance of equations of motion will break down.

Restricting ourselves for simplicity of presentation to the gravitational interaction of two bodies, the Sun and the Earth with $M_s >> m_e$, the equations of Earth's motion around the Sun are as follows:

$$m_e d^2x_i/dt^2 = Gm_eM_sx_i/(x_1^2 + x_2^2 + x_3^2)^{3/2},\ i = 1, 2, 3 \qquad (3.1)$$

Since $M_s = (4\pi/3)\rho R_s^3$, the equations (3.1) can be presented in a form that makes scaling invariance apparent:

$$d^2(x_i/R_s)/dt^2 = (4\pi/3)\rho G(x_i/R_s)/[(x_1/R_s)^2 + (x_2/R_s)^2 + (x_3/R_s)^2]^{3/2},\ i = 1, 2, 3 \qquad (3.2)$$

It is obvious that scaling invariance of equations of motion will hold in the general case of ***N*** gravitating bodies as well.

It is important to note that scale invariance of density $\rho$ is maintained because scale transformation is applied at all levels of fractal structure of matter in the universe where there is no such thing as elementary particle. The fact that electron has a spin (it rotates!) suggests that it is not an 'elementary' particle – it has some kind of structure. If electron has a structure, it doesn't seem reasonable to expect that the elements of that structure would not have structures in turn, *etc. ad infinitum*.

Now, if we take relativity of scale as universal principle of nature on par with relativity of motion, the inverse square law of gravitational interaction follows as its logical implication.

## CONCLUSION

Newtonian mechanics in its entirety is an inductive implication of what we call extended principle of relativity – Galilean relativity augmented with relativity of scale.

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